Chapter 5.3: Properties of Logarithms

In this section, we will go back and revisit the properties of exponents, to give us an insight into how to handle logarithms.

Logarithms give exponents as their answers, so…
the properties of logarithms are the same as the properties of exponents!

For example, when we **multiply** exponential expressions with the same base, we combine by **adding** the exponents: $x^a \cdot x^b = x^{a+b}$.

Property 1: Similarly, when we multiply inside logarithms, we can change this to **adding** the individual logarithms: $log_x(mn) = log_x m + log_x n$

EXAMPLES

a. To see how the above property works in practice, let's use our easiest base, 10, which we can also get on the calculator. Let *m* = 100 and *n* = 1000 in the property, above, and we get:

$$log_{10}(100 \cdot 1000) = log_{10}100 + log_{10}1000$$

Is this true?
Let's check using our calculators and what we know about logarithms.

$log_{10}(100 \cdot 1000)$	
$log_{10}(100{,}000)$	STEP 1: combine inside parenthesis
= 5	STEP 2: use your calculator to find the log This answer makes sense because $10^5 = 100{,}000$
$log_{10}100 = 2$	Use your calculator or the fact that $10^2 = 100$
$log_{10}1000 = 3$	Use your calculator or the fact that $10^3 = 1000$

Put it all together to get:

$$log_{10}(100 \cdot 1000) = log_{10}100 + log_{10}1000$$

$$log_{10}(100{,}000) = log_{10}100 + log_{10}1000$$

$$5 = 2 + 3$$

Yes, this is true! $5 = 5$

For exponential expressions with the same base, when we ***divide,*** we combine by **subtracting** the exponents: $\frac{x^a}{x^b} = x^{a-b}$.
Before you turn the page, use what you know about exponents to see if you can figure out Property 2 of logarithms. That is, $log_x\left(\frac{m}{n}\right) =?$

Property 2: Similarly, when we **divide** inside logarithms, we can change this to **subtracting** the individual logarithms: $log_x\left(\frac{m}{n}\right) = log_x m - log_x n$

b. To see how the above property works in practice, let's again use our easiest base, 10. Let *m* = 1000 and *n* = 10 in the property, above, and we get:

$$log_{10}\left(\frac{1000}{10}\right) = log_{10}1000 - log_{10}10$$

Is this true?

Let's check using our calculators and what we know about logarithms.

$log_{10}\left(\frac{1000}{10}\right)$

$log_{10}100$	STEP 1: combine inside parenthesis
= 2	STEP 2: use your calculator to find the log
	This answer makes sense because $10^2 = 100$
$log_{10}1000 = 3$	Use your calculator or the fact that $10^3 = 1000$
$log_{10}10 = 1$	Use your calculator or the fact that $10^1 = 10$
	This is also one of our identities, $log_b b = 1$

Put it all together to get:

$$log_{10}\left(\frac{1000}{10}\right) = log_{10}1000 - log_{10}10$$

$$log_{10}100 = log_{10}1000 - log_{10}10$$

$$2 = 3 - 1$$

Yes, it works! $\quad 2 = 2$

The final property involves what happens when we have an exponent inside a logarithm. This is similar to having an exponent raised to a power: $(x^a)^b = x^{ab}$, in which case we **multiply** the exponents.

Property 3: When we have an exponent inside a logarithm, we multiply the exponent by the individual logarithm: $log_x(m^n) = nlog_x m$

This one looks a little different, in that the n comes down in front and is not inside a logarithm. Let's see why in our next example.

c. If we let m= 100 and n = 3 in the above property, we get

$$log_{10}(100^3) = 3log_{10}(100)$$

$log_{10}(100^3)$

$log_{10}1{,}000{,}000 = 6$	STEP 1: combine inside parenthesis
	STEP 2: use your calculator to find the log
	(Also $10^6 = 1{,}000{,}000$)
$log_{10}100 = 2$	Use your calculator or the fact that $10^2 = 100$

Put it all together to get:

$$log_{10}(100^3) = 3log_{10}(100)$$
$$log_{10}(1{,}000{,}000) = 3log_{10}(100)$$
$$6 = 3\ (2)$$
Yes, it works! $6 = 6$

The table below summarizes the properties of exponents with the properties of logs.

Properties of Exponents and Logarithms

Property of Exponents	Words	Property of Logarithms	Words
$\boldsymbol{x^a \cdot x^b = x^{a+b}}$	When we **multiply** exponential expressions with the same base, we **add** each exponent.	$log_x(mn) = log_x m + log_x n$	When we **multiply** inside logarithms, we can change this to **addition** of the individual logarithms.
Example 1: $\boldsymbol{x^2 \cdot x^3 = x^{2+3=5}}$ Example 2: $\mathbf{10^2 \cdot 10^3 = 10^{2+3=5}}$		Example: $log_{10}(100 \cdot 1000) = log_{10}100 + log_{10}1000$ $log_{10}(100{,}000) = 2 + 3$ $5 = 5$	
$\boldsymbol{\frac{x^a}{x^b} = x^{a-b}}$	When we **divide** exponential expressions with the same base, we **subtract** each exponent.	$log_x\left(\frac{m}{n}\right) = log_x m - log_x n$	When we **divide** inside logarithms, we can change this to **subtraction** of the individual logarithms.
Example 1: $\boldsymbol{\frac{x^3}{x^1} = x^{3-1=2}}$ Example 2: $\mathbf{\frac{10^3}{10^1} = 10^{3-1=2}}$		$log_{10}\left(\frac{1000}{10}\right) = log_{10}1000 - log_{10}10$ $log_{10}(100) = 3 - 1$ $2 = 2$	
$\boldsymbol{(x^a)^b = x^{ab}}$	When we have an exponent raised to a power, we multiply the exponents.	$log_x(m^n) = nlog_x m$	When we have an exponent inside a logarithm, we bring the exponent down **in front of the log** and **multiply** it by the individual logarithm.
Example 1: $\mathbf{(x^2)^3} = \boldsymbol{x^{2(3)=6}}$ Example 2: $\mathbf{(10^2)^3 = 10^{2(3)=6}}$		$log_{10}(100^3) = 3log_{10}(100)$ $log_{10}(1{,}000{,}000) = 3log_{10}(100)$ $6 = 3\ (2)$ $6 = 6$	

Now let's use these properties to expand or condense logarithms.

Tip: If you are ***expanding*** logarithms, look for multiplication, division, or exponents, which you will be turning into addition, subtraction or multiplication of separate logs!

EXAMPLES – Expand each logarithm.

a. Expand $log_5(2ab)$.

$log_5(2ab) = log_5(2 \cdot a \cdot b)$
$= log_5 2 + log_5 a + log_5 b$

We can expand multiplication inside logarithms by rewriting as sums of *separate logs* (property 1). Since we have three separate products, so we *add* three separate logs.

b. Expand $log_2\left(\frac{3}{k}\right)$.

$= log_2 3 - log_2 k$

We can expand quotients (division) by rewriting as separate, *subtracted* logarithms (property 2). Since we had two things being divided, we end up with *two* separate logs.

c. Expand $log_b x^4$.

$= 4log_b x$

For this problem, we use property 3. Since we have an Exponent inside a log, we bring the exponent down in front of the logarithm and **multiply**.

Now let's continue the examples, but combining the properties together, and bringing in our rational exponents.

d. Expand $log_2 a^4 b^5$.

Notice that this example has both products (multiplication) and exponents.

$= log_2 a^4 + log_2 b^5$

Let's begin by using property 1 to separate the products into added terms.

$= 4log_2 a + 5log_2 b$

Next, use property 3 to rewrite the exponents as multiplication in front of each log.

e. Expand $log_4\left(\frac{5a}{\sqrt{b}}\right)$.

$= log_4 5 + log_4 a - log_4 \sqrt{b}$

Notice that this example has multiplication (5a is 5•a) **and** division, so we use property 1 **and** 2 to rewrite the multiplication and division as addition and subtraction.

It looks like we are done, right? But there is one more thing to do! Remember that we can express radicals as exponents! So even though it doesn't look like it, there is a **hidden** exponent here. See if you can figure out what to do with it before you turn the page!

Recall: $\sqrt{b} = b^{\frac{1}{2}}$.

So, we get $log_4\left(\frac{5a}{\sqrt{b}}\right) = log_4 5 + log_4 a - log_4\sqrt{b} = log_4 5 + log_4 a - log_4 b^{\frac{1}{2}}$

$= log_4 5 + log_4 a - \frac{1}{2} log_4 b$

f. $log_b \sqrt[4]{\frac{x^3 y}{z}}$

Recall that $\sqrt[4]{b} = b^{\frac{1}{4}}$ and begin by rewriting the 4th root as an exponent.

$log_b \sqrt[4]{\frac{x^3 y}{z}} = log_b \left(\frac{x^3 y}{z}\right)^{\frac{1}{4}}$

Next, before we expand the logarithm, remember that when we raise a power to a power, we multiply exponents, so let's rewrite the exponents first.

$log_b \left(\frac{x^3 y}{z}\right)^{\frac{1}{4}} = log_b \left(\frac{x^{3\bullet\frac{1}{4}} y^{\frac{1}{4}}}{z^{\frac{1}{4}}}\right)$

To multiply $3\bullet\frac{1}{4}$, multiply $\frac{3}{1}\bullet\frac{1}{4} = \frac{3}{4}$.

$= log_b \left(\frac{x^{\frac{3}{4}} y^{\frac{1}{4}}}{z^{\frac{1}{4}}}\right)$

Now expand, remembering that multiplication becomes addition of the separate logs and division becomes subtraction of the separate logs.

$= log_b\left(x^{\frac{3}{4}}\right) + log_b\left(y^{\frac{1}{4}}\right) - log_b\left(z^{\frac{1}{4}}\right)$

Finally, bring the exponents down in front of each log to multiply.

$$= \frac{3}{4} log_b x + \frac{1}{4} log_b y - \frac{1}{4} log_b z$$

g. Expand $log_a\left(\frac{2x^3}{3y}\right)$.

We first rewrite the multiplication and division as addition and subtraction:

$$log_a\left(\frac{2x^3}{3y}\right) = log_a 2 + log_a x^3 - (log_a 3 + log_a y)$$

But notice the parenthesis around the last two terms, because they are **both** in the numerator, thus **both are being divided, which becomes subtraction of each.** Distributing the subtraction, we get:

$$log_a 2 + log_a x^3 - log_a 3 - log_a y$$

We could have written it this way in the first place, knowing that both are being divided, so we must subtract *each* separate log.

Finally, we bring any exponents in front of their logs to multiply:

$$log_a 2 + 3 log_a x - log_a 3 - log_a y$$

EXAMPLES – Condense each logarithm by expressing as a SINGLE logarithm

Now, we are going backwards from the way we went before!

Tip: If you are ***condensing*** logarithms, look for addition, subtraction or multiplication which you will be turning into multiplication, division, or exponents in one single logarithm!

a. Express as a single logarithm: $log_m x - log_m y$

We have subtraction of two separate logs, so if we want to combine back into a single logarithm, this subtraction becomes division:

$log_m x - log_m y = log_m \left(\frac{x}{y}\right)$

b. Express as a single logarithm: $2log_a x + \frac{1}{3} log_a y$

First notice that we have constant multipliers in front of each term. These numbers that are multiplying in front of the separate logs can be rewritten as exponents inside the logs.

$$2log_a x + \frac{1}{3} log_a y = log_a x^2 + log_a y^{\frac{1}{3}}$$

We have addition of two separate logs, so if we want to combine back into a single logarithm, this addition becomes multiplication:

$$= log_a (x^2\, y^{\frac{1}{3}})$$

$$= log_a (x^2\, \sqrt[3]{y}) \qquad \text{because } y^{\frac{1}{3}} = \sqrt[3]{y}$$

c. Express as a single logarithm: $2log_3 x - 3log_3 y - log_3 z$

We have constant multipliers in front of two of the terms. These numbers can be rewritten as exponents inside the logs.

$$= log_3 x^2 - log_3 y^3 - log_3 z$$

Now, we have *two* divisions, or quotients. BOTH of these will end up together in the denominator. WHY? Well, we can think of the expression as

$$= log_3 x^2 - (log_3 y^3 + log_3 z)$$

or we can remember that a negative logarithm, like a negative exponent, is a reciprocal. So, both negative terms go in the denominator.

$$= log_3 \frac{x^2}{y^3 z}$$